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The time dimension of parking economics

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ABSTRACT

A model of demand for parking, evolving over time, is proposed. The model features both extensive (whether to park) and intensive (for how long to park) margins of parking demand, allows multidimensional heterogeneity of parkers, and evolution of demand throughout the day. I show that the optimal price for parking is proportional to the rate of arrival of new parkers and is inversely related to the square of the occupancy rate, which is different from previously discussed pricing methods. I show that the primary purpose of pricing is to regulate departures, rather than arrivals, of parkers. I also find that asymmetric information about parkers' characteristics does not prevent the parking authority from achieving the social optimum. A numerical example compares the optimal policy against the alternatives.

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1. Introduction

Starting from Vickrey (1954), economic theory in general, and multiple studies in particular, prescribe to tackle the problem of curbside parking congestion with price regulation. How exactly pricing affects parking, however, remains an open question. Most existing research asserts that a higher parking price either reduces the number of motorists traveling to the area in question (Ahmadi Azari et al., 2013; Arnott and Inci, 2006; van Ommeren and Russo, 2014), or affects the location of parking (Anderson and de Palma, 2004; Qian et al., 2012), or both (Arnott, 2014; Arnott and Rowse, 1999; Li et al., 2007; Madsen et al., 2013). In this paper, I highlight another channel through which parking prices reduce congestion, namely the duration of parking period, or the *intensive margin* of parking demand. The importance of this channel of parking regulation has been highlighted in popular writings on the topic. For example, Shoup(2005, p. 363) argues that higher parking rates, by reducing the duration of parking (intensive margin), may increase availability of parking to the point that the number of travelers (extensive margin) actually increases. This paper develops a model of parking rates. The model features time-varying intensity of arrival of the motorists, and calculates the socially optimal parking price schedule.

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In a world where parking rates not only deter entry of new motorists, but also expedite exit of those already parked, the philosophy of optimal parking rates is different. The existing research on optimal parking rates, focused on the extensive margin of parking demand, views parking pricing as analogous to that of road congestion pricing: wherever a road usage is below its capacity, motorists have no externality on each other and should travel for free; when full capacity is reached, a negative externality of vehicle use arises, and a toll should be optimally imposed. The existing research on parking (e.g., Shoup, 2005) is very similar as it prescribes to target a specific usage/occupancy rate. The only difference is that the recommended occupancy target is less than 100% (the most popular figure is 85%), so that few spaces are left open for newly arriving parkers. Arnott (2014), in a model with exogenous parking duration, shows that the optimal occupancy may vary over time, with (extensive margin of) demand for parking.

I argue that parking congestion should *not* be viewed as analogous to road congestion. A hundred of motorists on the same road create congestion and reduce each other's welfare; a hundred of motorists already parked do *not* reduce each other's welfare. More generally, *the motorists already parked do not compete with each other for space. The only conflict of interest out there is between a motorist already parked and a motorist still searching for a place to park.² Therefore, the right parking price, that aims to expedite or delay the departure of already parked vehicles, should explicitly take into account not only the occupancy rate (how many are parked) but also the rate of new arrivals (how many are looking for parking). At the same occupancy rate, a higher arrival (=turnover) rate implies a greater negative externality of already parked vehicles, thus should imply a higher price for parking.*

Despite the simplicity of this argument, it has been surprisingly overlooked by policy makers. For example, the federally funded SFpark project of San Francisco, considered to have employed the state-of-the art parking pricing policy, does not even keep the data on the frequency of new arrivals despite an infinitesimal additional cost of doing so,³ and bases its prices solely on the occupancy data.

Many aspects of the economics of urban parking have been analyzed. Glazer and Niskanen (1992), the only known existing model of the intensive margin of parking demand, points out that a higher parking price per unit of time, by increasing turnover of parkers, may increase traffic. Zou et al. (2015) use the mechanism design approach to optimally allocate heterogenous parking spots to heterogenous motorists prior to their arrival, using modern communication technology. Boyles et al. (2015) explicitly models the geography of the search for a parking spot. Fosgerau and de Palma (2013) study how parking fees may substitute tolls on congested roads; in their model however parking space *per se* is unlimited, so there is no search for parking. Inci (2014) contains the most recent review of the parking literature.

A large strand of literature has analyzed the phenomenon known as "cruising for parking", essentially a negative externality of those who search for parking on through traffic. Arnott and Inci (2006) is an early theoretical contribution, while Shoup (2006) provides empirical estimates of the scale of the problem. More recently, Geroliminis (2015), Cao and Menendez (2015), Zheng and Geroliminis (2016) provide further theoretical elaborations on the topic. The current paper, however, does not address the problem of cruising for parking, because this problem is likely to exist only in case of suboptimal (too low) price for parking. When pricing is optimal, the calibration exercise of this paper shows that, during the peak arrivals period, the average duration of search for a parking spot is only 29 s, which essentially eliminates the cruising problem. Outside of the rush hour, the search time may be longer, but the number of vehicles searching is small, so the aggregate negative externality of the searchers is small.

2. Model

Consider the world with continuous time and a continuum of motorists. The length of a day is *T* units of time, and all days are identical. That is, the equilibrium value of a model parameter at time *t* is equal to that of time t + T. Motorists differ by their type $v \in V$, which may have one or more dimensions. Motorists of different types have different demand for parking, detailed below. The motorists appear in the model exogenously; the rate of appearance of a type-v motorist at time *t* is A(t, v). Immediately after appearance, a motorist decides whether to start searching for parking, or to quit the model permanently.⁴ Those who quit enjoy the outside utility of $U_0(v)$. Those who start searching are referred to as *searchers*. Search is a Poisson process in which exogenously supplied parking sites are randomly sampled at rate *r*. At each instance of sampling, a searcher randomly selects a parking site from the mass *N* of all existing sites. If the site is vacant, the searcher occupies it, the search is over, and the parking period begins. Otherwise, the search continues. The cost of search is *c* per

² One may argue that two motorists simultaneously searching for parking also have an externality on each other. Such externality, however, is only *prospective*: the negative impact that searcher A has on searcher B is conditional on the fact that A finds a spot before B, becoming a parker, and that B stumbles upon the site occupied by A before finding his own vacant site. An optimal policy need not account for this prospective externality if the immediate externality of parkers on searchers is properly addressed.

³ The installed hardware does measure the precise time and location of arrivals, but the program operators choose not to keep this data, in order do save few hundred megabytes of computer memory per year.

⁴ Thus, the model is missing one more margin of demand for parking: the decision when to start searching for parking. Introducing such margin would somewhat smooth the extensive margin (number of arrivals) over time, but I believe it would not change the main results qualitatively. Fosgerau and de Palma (2013) model this margin of demand explicitly.

unit of time. We assume that the search cost is a purely private cost. The search must continue until a vacant parking site is found.⁵ Note that the search technology is the same for motorists of all types.⁶

The motorists already parked are referred to as *parkers*. For a type- ν motorist/parker, the marginal value of being parked for an additional unit of time is $u(\tau, \nu)$, where τ is the time elapsed since the beginning of parking period. Without loss of generality, $u(\tau, \nu)$ does not depend on the time of appearance *t*: the time dimension can be incorporated into the set of types \mathcal{V} . We assume that $u(\tau, \nu)$ is both continuous and strictly decreasing with respect to τ , i.e., every additional minute of parking is less valuable than the previous one. The function *u* is assumed to be deterministic, i.e., every parker knows exactly his value of parking at every future time. Adding an idiosyncratic stochastic component would complicate the model, but would have no effect on aggregate parameters and thus the main theoretical results would be preserved.

We also assume that, for each type v, there exists a finite $\tau^{\max}(v)$ such that $u(\tau, v) < 0$, $\forall \tau \ge \tau^{\max}(v)$. In other words, every particular type of parkers will eventually depart even in the absence of parking regulation.

A parker may permanently exit the parked mode, and the model altogether, at any moment of time. There is zero additional value gained after the departure.

Define by $I(t, v) \in [0, 1]$ the endogenous probability that a type-v motorist, who appeared in the model at time t, chooses to enter the search mode. Define by S(t, v) the stock of searchers of type v at time t, and by q(t) the occupancy rate, i.e., the share of occupied parking sites.

For an infinitesimal period of time [t; t + dt], the probability that a searcher samples a parking site is rdt; the probability that such site is vacant is 1 - q(t). Therefore, the exit rate from the search mode is r(1 - q(t)), the mass of type-v parkers exiting the search mode and beginning their parking sessions is

$$R(t, v) = r(1 - q(t))S(t, v),$$
(1)

and the evolution of the stock of searchers is described by

$$S(t, v) = I(t, v)A(t, v) - R(t, v).$$

The social cost of search at time *t* is equal to $c \int_{\mathcal{V}} S(t, v) dv$.⁷

Define by $\tau^+(t, v)$ the total parking duration of a type-v motorist who began her session at t. Thus, she remains parked within $[t, t + \tau^+(t, v)]$. Define by $\tau^-(t, v)$ the total parking duration of a type-v motorist who ends his session at t. Thus, he remains parked within $[t - \tau^-(t, v), t]$. We assume that $\tau^-(t, v)$ is unique for all t, v; such assumption is needed primarily for notational simplicity of the subsequent analysis.⁸

By definition, $t = t' + \tau^+(t', v)$ iff $t' = t - \tau^-(t, v)$, $\forall t, v$.

With new notations, we can present the occupancy rate as

$$q(t) = \frac{1}{N} \int_{\mathcal{V}} \int_{0}^{\tau^{-}(t,\nu)} R(t-\tau,\nu) d\tau d\nu.$$
(3)

The social planner's objective is to maximize the social welfare aggregated across all motorists that appear in the model during a day:

$$V = \int_0^T \int_{\mathcal{V}} \int_0^{\tau^-(t,\nu)} u(\tau,\nu) R(t-\tau,\nu) d\tau d\nu dt - c \int_0^T \int_{\mathcal{V}} S(t,\nu) d\nu dt + \int_0^T \int_{\mathcal{V}} (1-I(t,\nu)) A(t,\nu) U_0(\nu) d\nu dt,$$
(4)

using $I(\cdot, \cdot)$ and $\tau^{-}(\cdot, \cdot)$ as controls, and subject to (1), (2), (3), and to the condition of time cyclicality.

3. Simplified model

Analyzing the optimal parking price schedule in the general case is a non-standard dynamic optimization problem that involves solving differential equations with endogenously varying time lags. The general solution may be impossible to analyze analytically, and we proceed by developing a simplified version of the model.

The simplification is based on the presumption that the duration of search for parking is short enough so the change of aggregate conditions (price, occupancy rate) faced by motorists during search is negligible. Is the duration of search indeed that short? Shoup (2006) estimates it to be between 3.5 and 14 min in actual (typically underpriced and thus congested) conditions. Under the optimal policy, it is likely to be far smaller. I conclude that, while the simplification may render the model unrealistic in case of arrival/departure disruptions (e.g., large number of motorists arriving to a port shortly before

(2)

⁵ It is theoretically possible that, if aggregate conditions change during one's search process, a searcher decides to quit before finding a vacant spot. Section 3, by assuming that a typical search is short enough, and aggregate conditions are smooth enough over time, removes such possibility from the model.

⁶ It is possible to generalize the model to heterogenous search cost *c*. Then, while individual decisions would depend on individual *c*, the socially optimal price for parking would depend on *c* averaged across those actively looking for parking.

⁷ Millard-Ball et al. (2014) scrutinize the relationship between occupancy and the search process in more detail.

⁸ Multiple $\tau^-(t, v)$ exist when parkers of the same type v, but with different arrival times, depart at the same time t. Since the utility from parking is continuous with respect to duration, multiple $\tau^-(t, v)$ can occur only if the price per minute of parking discontinuously increases at time t. While theoretically possible, such policy is never optimal and so is omitted.

a ferry departure), it is useful in more usual scenarios such as downtown parking. Geroliminis (2015) makes a very similar assumption. With this assumption in place, the expected cost of search for parking is $\frac{c}{r(1-a(t))}$.

I now replace the original model with the simplified one, defined below, that makes calculation of optimal parking rates straightforward.

Suppose parkers appear at rate A(t, v) and decide whether to participate in the parking process, as in the original model. However, those who decided to park, instead of searching, park instantly. Each parking motorist must pay the expected search cost from the original model, $\frac{c}{r(1-q(t))}$. This payment is not received by anyone, but is a deadweight loss. Otherwise, the alternative model is identical to the original one.

In the simplified model, since the stock of searchers is always empty, exit from the search mode R(t, v) is equal to entry into the search mode I(t, v)A(t, v), thus the occupancy rate can be redefined by (cf.(3))

$$q(t) = \frac{1}{N} \int_{\mathcal{V}} \int_{0}^{\tau^{-}(t,\nu)} I(t-\tau,\nu) A(t-\tau,\nu) d\tau d\nu.$$
(5)

3.1. Planner's problem

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We can redefine the social planner's objective as to maximize (cf.(4))

$$V = \int_{0}^{T} \int_{\mathcal{V}} \int_{0}^{\tau - (t,v)} I(t - \tau, v) A(t - \tau, v) u(\tau, v) d\tau dv dt$$

- $\frac{c}{r} \int_{0}^{T} \frac{1}{1 - q(t)} \int_{\mathcal{V}} I(t, v) A(t, v) dv dt + \int_{0}^{T} \int_{\mathcal{V}} (1 - I(t, v)) A(t, v) U_{0}(v) dv dt.$ (6)

The first component of (6) describes the total welfare gain from the parking process; the second component is the total welfare loss from the search process; the last component is the total welfare gain from enjoying the outside opportunity. The social planner's controls are the trajectories $I(\cdot, v)$ and $\tau^-(\cdot, v)$ (or, interchangeably, $\tau^+(\cdot, v)$) for every v, subject to (5).

3.1.1. Optimal duration of parking

The first-order condition of optimal $\tau^{-}(t, v)$ is

$$\frac{\partial V}{\partial \tau^{-}(t,v)} = I(t - \tau^{-}(t,v),v)A(t - \tau^{-}(t,v),v)u(\tau^{-}(t,v),v) - \frac{c}{rN}\frac{I(t - \tau^{-}(t,v),v)A(t - \tau^{-}(t,v),v)}{(1 - q(t))^{2}}\int_{\mathcal{V}}I(t,v')A(t,v')dv' \begin{cases} = 0, & \tau^{-}(t,v) > 0, \\ \leq 0, & \tau^{-}(t,v) = 0. \end{cases}$$
(7)

If $I(t - \tau^{-}(t, v), v)A(t - \tau^{-}(t, v), v) = 0$, the problem is meaningless because the parkers of type v simply do not enter the system at time $t - \tau^{-}(t, v)$, and therefore their duration of parking has no effect on social welfare. If $I(t - \tau^{-}(t, v), v)A(t - \tau^{-}(t, v), v) > 0$, we can redefine (7) as

$$u(\tau^{-}(t,\nu),\nu) \le \frac{c}{rN} \frac{\int_{\mathcal{V}} I(t,\nu')A(t,\nu')d\nu'}{(1-q(t))^{2}},$$
(8)

with equality if $\tau^{-}(t, v) > 0$. Note that the right-hand side of (8) is the same for all types $v \in \mathcal{V}$; denote it by

$$p(t) \equiv \frac{c}{rN} \frac{\int_{\mathcal{V}} I(t, v') A(t, v') dv'}{(1 - q(t))^2}.$$
(9)

It is equal to the externality that a parker at time *t* has on the searchers. Intuitively, such externality must be proportional to the number of searchers (the integral in (9)). The externality on each searcher $\frac{c}{rN} \frac{1}{(1-q)^2}$ is the derivative of the search cost $\frac{c}{r} \frac{1}{1-q}$ with respect to *q*, times the impact of a given parker on aggregate occupancy, $\frac{1}{N}$.

The optimal duration of parking satisfies

$$\tau^{-}(t,v) = u^{-1}(p(t),v), \tag{10}$$

where $u^{-1}(u(x, v), v)$ solves (cf.(8)) $u(u^{-1}(p(t), v), v) \equiv p(t)$. Since the solution to (8) is not necessarily unique, we assume $u^{-1}(p, v)$ always takes the value that corresponds to the global maximum of social welfare with respect to $\tau^{-}(t, v)$.

We can now relate q(t) to p(t) by modifying (5):

$$q(t) = \frac{1}{N} \int_{\mathcal{V}} \int_{0}^{u^{-1}(p(t),\nu)} I(t-\tau,\nu) A(t-\tau,\nu) d\tau d\nu.$$
(11)

The equations (9) and (11) constitute a system that jointly determines p(t) and q(t). The social optimum at time t is visualized on Fig. 1. The downward-sloping curve represents the demand for parking (11). The upward-sloping curve is the social cost of parking (i.e., the externality on motorists arriving at time t) as described by (9). This curve depends on q(t) directly, as well as indirectly via endogenous arrival decisions I(t, v): a higher occupancy q(t) increases the search cost and thus reduces arrivals. The intersection of the two curves determines the socially optimal occupancy.



Fig. 1. Occupancy and shadow price.

What is the socially optimal relationship between the rate of arrival of new parkers, $f_{y}I(t, v)A(t, v)dv$, and the occupancy rate? Arnott (2014), in a model without the intensive margin of parking demand, prescribes a positive relationship. Our model prescribes just the opposite: if fewer searchers appear in the model, the shadow price of parking is lower, and it is optimal to keep more spaces occupied. Visually, a lower appearance rate A(t, v) shifts the social cost curve on Fig. 1 downward, causing the equilibrium occupancy rate to increase and the price to decrease. This finding is consistent with Martens et al. (2010) who, in a geospatial model of search for parking, recommend to increase the occupancy rate target from the widely accepted level of 85%, especially in case of relatively low parking turnover levels.⁹

3.1.2. Optimal entry decision

The first-order condition of optimal I(t, v) in the maximization problem (6) is

$$\frac{\partial V}{\partial I(t,v)} = A(t,v) \int_0^{\tau^+(t,v)} u(\tau,v) d\tau - \frac{c}{r} \frac{A(t,v)}{1-q(t)} - \frac{c}{r} \frac{A(t,v)}{1-q(t)} \int_0^{\tau^+(t,v)} \int_0^{\tau^+(t,v)} \int_v^{\tau^+(t,v)} \frac{I(t+\tau,v')A(t+\tau,v')}{(1-q(t+\tau))^2} dv' d\tau - A(t,v) U_0(v) \begin{cases} \ge 0, & I(t,v) = 1, \\ = 0, & I(t,v) \in (0,1), \\ \le 0, & I(t,v) = 0. \end{cases}$$
(12)

In (12), the first component is the welfare gained from the process of parking; the second component is own cost of search; the third component is the additional cost of search for subsequent motorists, caused by elevated occupancy; the fourth component is the foregone outside opportunity. If A(t, v) = 0, the motorists of type-v do not appear at t and their decisions are immaterial; otherwise, it is socially optimal to participate in the parking process if

$$\int_{0}^{\tau^{+}(t,\nu)} u(\tau,\nu) d\tau - \frac{c}{r} \frac{1}{1-q(t)} - \frac{c}{rN} \int_{0}^{\tau^{+}(t,\nu)} \int_{\mathcal{V}} \frac{l(t+\tau,\nu')A(t+\tau,\nu')}{\left(1-q(t+\tau)\right)^{2}} d\nu' d\tau \ge U_{0}(\nu), \tag{13}$$

with I(t, v) = 1 in case of strict inequality.

3.2. Optimal regulation of parking

A motorist already parked makes the decision to exit by comparing her marginal utility of parking $u(\tau, v)$ against a marginal cost of parking. Since there is no "natural" marginal cost, it consists entirely of a price per unit of parking time imposed by the planner. Thus, the planner regulates exit from parking by imposing a price $\tilde{p}(t, v)$, such that motorists of type v that are parked between t and t + dt pay $\tilde{p}(t, v)dt$.

A motorist considering whether to search for parking compares the expected value of parking, integrated over the entire (optimally chosen) parking period, less the search costs and the monetary costs, against the outside opportunity. Therefore, the planner can regulate the entry process by imposing, on top of the price $\tilde{p}(t, v)$, an entry fee f(t, v) paid by motorists of type v entering the parking session at time t once, at the time of entry. Such two-part tariff is sufficient to regulate both margins, extensive and intensive, of parking demand.

With such two-part tariff in place, the motorists of type v appearing at t' choose whether to park, and for how long to park, by maximizing the following expected utility:

$$U(t',\nu) = I(t',\nu) \int_0^{\tau^+(t',\nu)} u(\tau,\nu) - \tilde{p}(t'+\tau,\nu) d\tau - \frac{c}{r} \frac{I(t',\nu)}{1-q(t')} - I(t',\nu)f(t',\nu) + (1-I(t',\nu))U_0(\nu)$$
(14)

⁹ Italics are mine.

3.2.1. Duration of parking

The duration of parking for those arriving at t' is relevant only if I(t', v) > 0; the corresponding first-order condition is

$$u(\tau^+(\tau, v), v) \le p(\tau + \tau^+(\tau, v), v),$$
(15)
the equality if $\tau^+(t', v) > 0$. Using the mutual dependence of $\tau^+(\cdot, v)$ and $\tau^-(\cdot, v)$, and defining $t = t' + \tau^+(t', v)$, we can

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with equality if $\tau^+(t', v) > 0$. Using the mutual dependence of $\tau^+(\cdot, v)$ and $\tau^-(\cdot, v)$, and defining $t \equiv t' + \tau^+(t', v)$, we can rewrite (15) as

$$u(\tau^{-}(t,\nu),\nu) \le \tilde{p}(t,\nu),\tag{16}$$

with equality if $\tau^{-}(t, v) > 0$. To make the duration of parking socially optimal, it must satisfy (10); substituting the latter into (16), for positive parking durations,

$$\tilde{p}(t,\nu) = p(t), \forall t,\nu.$$
(17)

We arrive at the following

Theorem 1. The optimal price for parking is proportional to the rate of arrival of new parkers, and is inversely related to the square of the vacancy rate.

For proof, observe the definition of p(t) in (9). Note that the recommendation of Theorem 1 is quite different from currently practiced pricing methods, in particular from the policy of setting the price to target a constant occupancy rate q(t) (e.g., within the SFpark project).

3.2.2. Entry decision

Suppose the price is set at the socially optimal level, and therefore parking durations are socially optimal, as well. We now analyze which entry fee f(t, v) induces the socially optimal decision to participate in search for parking. The motorist of type v arriving at t chooses to search for parking if

$$\int_{0}^{\tau^{+}(t,\nu)} \left[u(\tau,\nu) - p(t+\tau) \right] \mathrm{d}\tau - \frac{c}{r} \frac{1}{1-q(t)} - f(t,\nu) \ge U_{0}(\nu).$$
(18)

To achieve socially optimal participation decisions, (13) must also be true. Since the right-hand sides of (18) and of (13) are the same, it is sufficient for the social optimum that the left-hand sides are the same, too. This implies that

$$f(t,v) + \int_0^{\tau^+(t,v)} p(t+\tau) d\tau = \frac{c}{rN} \int_0^{\tau^+(t,v)} \frac{1}{\left(1 - q(t+\tau)\right)^2} \int_{\mathcal{V}} I(t+\tau,v') A(t+\tau,v') dv' d\tau.$$
(19)

We can now formulate the following

Theorem 2. If parking departures are optimally regulated, there is no need to regulate parking arrivals.

For proof, use the definition of price (9) to observe that $\int_0^{\tau^+(t,v)} p(t+\tau)d\tau$ in (19) is equal to the right-hand side of (19), and therefore f(t, v) = 0, $\forall t, v$. In other words, the price of parking p(t), set to optimize parking departures, already helps to internalize all externalities that parkers have on searchers, and no additional pricing is needed. Again, this result contrasts with the existing theoretical literature on parking, which attempts to regulate the arrivals and typically ignores the departures regulation by assuming exogenous duration of parking.

Finally,

Theorem 3. If motorists' types are their private information, the first-best allocation can still be achieved.

This is due to the fact that the optimal prices set by the planner, $\tilde{p}(t, v) = p(t)$ and f(t, v) = 0, in fact do not depend on the motorists' types. This, in turn, is because the externality that a parker has on searchers (which is being internalized by regulation) does not depend on the parker's type; it only depends on the fact of the parker's presence at a given time. Arnott and Kraus (1998) elaborate this intuition in more detail.

4. Private supply of parking

This section investigates whether the social optimum can be achieved with privatized supply of parking space. In the presence of costly search for parking, every match of a motorist to a parking provider generates an economic surplus: if they do not reach agreement about the price for parking, the motorist has to incur additional search costs until another vacant spot is found, while the provider has to forego parking revenues until the next motorist arrives.

One way to divide the economic surplus is by prescription of an external authority. In this case, the socially optimal parking allocation of Section 3 can be achieved. If the regulated price for parking is the same for all parking spots at a given point in time, then every motorist that found an empty spot has no incentive to search for another spot, and every parking provider that has been visited by a motorist has no incentive to wait for another motorist. If the price is socially optimal as prescribed by Theorem 1, the allocation is socially optimal, as well.

If the price is not externally regulated, it depends on the relative bargaining power (the ability to influence price in their own favor) of the motorists and providers. The nature of the parking market is such that all bargaining power is likely to



Fig. 2. Motorist appearance over time.

belong to the providers: they can set the price well in advance, while the passing motorists have no physical ability to bargain about the price, and can only, literally, take or leave the offers they find.

With providers having full bargaining power, they will attempt to extract the entire welfare surplus from the motorists. If motorist types were observable, this would amount to setting the price equal to instantaneous utility from parking, $u(\tau, v)$. It is much more likely however that motorist types are their private information; in this case, the price must be the same for all types. The exact expression for the price remains to be determined, but it is obvious that it will not be socially optimal. For example, at the times than no new arrivals are expected, Theorem 1 states that the socially optimal price is zero; a profit-maximizing parking provider will set a strictly positive price, causing premature departure of some parkers.

To conclude, if the parking authority chooses to privatize parking sites, it may still have to regulate the price.

5. Numerical example

We illustrate the proposed pricing scheme with a numerical example. Suppose the parking capacity is N = 100 and the time interval is [0, T] = [0, 24]. The set of all types is unidimensional, $\mathcal{V} = [0, \infty)$. The appearance rate is characterized by $A(t, v) = B(t) \exp(-\frac{v}{2})$, $\forall v \in [0, \infty)$, where $\lambda = 10$. The marginal value of parking is

$$u(\tau, \nu) = \nu - s\tau, \tag{20}$$

with s = 2. Essentially, the type v is the instantaneous value at the beginning of parking session, and this instantaneous value decays linearly over time. The mean initial value is then equal to $\lambda = 10$, and the mean duration of unrestricted parking is $\frac{\lambda}{s} = 5$ h. The outside opportunity is assumed to be equal to zero, primarily for the purposes of better analytical tractability of the example: $U_0(v) = 0$. The B(t) is as follows:

Time, t	B(t)
0–5	0
5–9	2(t-5)
9–13	-2(t-13)
13-24	0

The evolution of B(t) over time is illustrated on Fig. 2. All motorists appear between 5 am and 1 pm, with a peak of B = 8 at 9 am. Such schedule of motorist appearance may not be the most empirically-driven, but useful to highlight the differences between regulation methods in the presence of time-varying demand for parking.

To calibrate the search rate r, suppose there is on average one parking site per 10 m of the curb space.¹⁰ A vehicle traveling at 15 kmh (a likely speed of a motorist searching for parking) may then inspect 1500 parking sites per hour, thus it is reasonable to assume r = 1500.

The time cost of searching *c* is worth a discussion. Recent theoretical literature on parking (e.g., Arnott and Rowse, 1999) emphasizes that, besides the search itself, there is another cost associated with the search for parking: the need to walk from the location of parking to the destination, and back to the location of parking. A motorist who searches for parking longer is likely to end up further away from the destination, and to spend more additional time on walking. Thus, there is a multiplier effect of the search process. To assess the magnitude of such multiplier, we need to introduce some assumptions about the geography of search. Suppose a motorist searches on a straight line, i.e., cannot make turns or uturns while searching. The search begins *a* units of time before arrival to destination, and continues until a parking site is found. The exit rate from the search state is r(1 - q) and is assumed to be constant for the duration of search, as explained in Section 3. Then, the expected driving time (at the speed of a searcher) from the parking location to the destination is $\int_0^{\infty} |t - a|r(1 - q) \exp(-tr(1 - q))dt = a + \frac{2}{r(1 - q)} \exp(-ar(1 - q)) - \frac{1}{r(1 - q)}$; its minimization with respect to *a* yields

¹⁰ Most vehicles actually require only half of that, but we account for the fact that the curb space is frequently utilized for other purposes, such as road junctions, pedestrian crossings, and fire hoses.



Fig. 3. Occupancy over time, under various regulation policies.

 $a^* = \frac{\log(2)}{r(1-q)}$, and the corresponding expected driving time is $\int_0^\infty |t - a^*| r(1-q) \exp(-tr(1-q)) dt = \frac{\log(2)}{r(1-q)}$. For example, with 99% occupancy, a motorist should start the search 0.0462 h, or 2 min 46 s before passing by the destination; the expected driving time from parking location to destination will also be 0.0462.

The time needed to walk this distance there and back is twice the ratio of the driving speed (in the search mode) to the walking speed. We assume this ratio to be equal to 4 (15 kmh vs. 3.75 kmh), thus the total time spent on search with walking is $\frac{1}{r(1-q)}(1+2 \times 4\log(2))$, compared to $\frac{1}{r(1-q)}$ spent on search *per se*. If the value of the search time is 10 dollars per hour (i.e., the mean initial value of parking), then the cost of search *plus walking* is $10 \times (1+2 \times 4\log(2)) \approx 65$ dollars per hour of search. Following this intuition, we assume c = 65.

One problem associated with the above logic is that the need to walk to and from the final destination increases not only the search cost but also the duration of parking, a phenomenon ignored in this paper. While ignoring this may lead to underestimation of occupancy during high-demand period, it should not compromise the comparison of various regulation methods between each other.

I calculate parking patterns under the following five regulation scenarios:

- 1. No regulation at all. The planner does not intervene the parking process. Parkers stay until zero marginal value of parking.
- The (optimal) time-invariant price for parking. This scenario is realized, for example, in the city of Moscow.¹¹ Parkers stay until the marginal value of parking equals the price.
- 3. Time-invariant parking time limit. To isolate the effects of time limits, I assume no parking fees are collected. The time limits are popular in the cities of Australia, although a fee usually does apply in addition. As detailed below, a finite *optimal* time limit does not exist in this setting, so I analyze an ad-hoc, non-optimal, time limit.
- 4. Price regulation targeting (optimal) time-invariant occupancy rate. There is a lower bound of zero on the price, i.e., motorists are not subsidized when occupancy falls below the target. This scenario has been implemented within the SFpark project in San Francisco.
- 5. The optimal price proposed by this paper, as defined in (9) and illustrated on Fig. 1.

For each scenario, I analyze and compare the following model parameters: occupancy rate, expected search time, entry value cutoff, price (if applicable), and social welfare generation. The technical details of the analysis are in the Appendix.

5.1. Results

The evolution of equilibrium occupancy over time, under various regulation policies, is shown in Fig. 3. Not surprisingly, the highest occupancy is achieved when there is no regulation at all; in our example, the maximum occupancy without regulation is 99.93%, i.e., one vacancy per almost 1500 parking sites; it is achieved at 8: 59, i.e., at the time of peak rate of motorist appearance.

¹¹ Although the optimality of the posted price has never been discussed.



Fig. 4. Expected search time over time, under various regulation policies.

The welfare-maximizing constant price was found to be P = \$4.83, or 48.3% of the mean initial value of parking. Since this regulation applies at all times of the day, it is not surprising that occupancy is also reduced by this policy at all times of the day. This regulation reduces congestion dramatically, with peak occupancy being 99.72% (one vacancy per 360 sites), at 10: 08.

The time-limit policy is worth special discussion. In the world of homogenous motorists, the time limit yields the same outcomes as the price regulation. For example, if the price is such that all motorists park for 4 h, such price can be replaced by the 4 h time limit with the same consequences for the aggregate welfare. However, when motorists are heterogenous as in our case, the time-limit policy results in poorer welfare outcomes. Some motorists are forced to depart while their value of parking is still high, while other motorists are allowed to stay while their value is already low. At the time of peak congestion, the time-limit policy leads to what I call the *adverse selection*, or crowding out of high-value parkers by low-value parkers. When there is no regulation at all, low-value motorists are kept out by the prospect of search; when there is price regulation, they are kept out by the prospect of monetary expenses. With time-limit policy, there is nothing that keeps them out so they fill nearly all parking spaces vacated by the regulation. As a result, the time-limit policy is least effective at the time when regulation is most needed, as demonstrated on Fig. 3. Such regulation may in fact be worse, from social welfare perspective, than no regulation at all. In particular, in our example I find that welfare-maximizing time limit is infinite, i.e., no time limit at all. For illustration purposes, I have calculated the equilibrium for a 3 h time limit, short enough to have any effect on peak occupancy. With such limit, the peak occupancy is 99.84% (one vacancy per 625 sites) at 8: 41. At the same time, the limit implies zero occupancy between 16:00 (3 h after arrivals end) and 5:00, when new arrivals begin.

The welfare-maximizing occupancy target was found to be equal to 95.99%, which corresponds to one vacant site out of 25, and which is somewhat higher than the 85% recommended by Shoup (2005).¹² Since this policy restricts behavior only during hours of peak demand, the off-peak occupancy is identical to that of no regulation. Note that peak occupancy lasts until 15:19, i.e., more than two hours after the end of the arrival period. As discussed in the Appendix, this is due to the fact that no one exits for some time after the end of the arrivals period.

Finally, the optimal occupancy exactly matches the intuition outlined in the introduction of this paper. During off-peak hours, it is nearly identical to that of no-regulation scenario. At the beginning of congestion period, the optimal occupancy is lower than that of occupancy-target scenario: in the presence of high arrival rate, it is optimal to keep more spaces open in order to reduce search costs for newly arriving motorists. On the contrary, at the end of congestion period, the optimal occupancy is higher than that under the occupancy target: in the presence of reduced arrival rate, there is less competition for vacant spaces, so it is optimal to allow those already parked to stay longer. Shortly before the end of the arrivals period at 13:00, departures halt, leading to 99.53% occupancy (one vacancy per 215 sites) at the end of the arrivals period and for some time beyond it.

¹² Such a high optimal rate may be due to the assumption of a continuum of motorists, and thus fully deterministic occupancy rates. Other research, e.g., Arnott (2014); Martens et al. (2010) analyze a more realistic case of discrete motorists and stochastic occupancy rates. The latter assess the optimal occupancy rate empirically, at 93%.



Fig. 5. Entry value cutoff over time, under various regulation policies.

Fig. 4 demonstrates the expected search time at different times of the day under different policies. It is equal to $\frac{1}{r(1-q(t))}$. Under the no-regulation scenario, the search time reaches nearly 60 min during the time of maximum congestion. Under the occupancy target policy, the expected search time during the congestion period is 60 s. Under the optimal policy, the expected search time day: at 9:00, the time of maximum arrivals, it is only 29 s; by 12:25 when departures halt, it is already 108 s; by the end of arrivals at 13:00, it reaches the maximum of 9 min. Such a high search cost is socially optimal because, by the end of the arrivals period, very few motorists have to pay this cost.

Fig. 5 demonstrates the entry value cutoff: only those whose initial value of parking is above the cutoff choose to search for parking. Therefore, it illustrates the extensive margin of parking demand: a higher cutoff implies a lower arrival rate. This cutoff is determined by the cost of search and, where applicable, by the monetary cost of parking.

With no regulation or time-limit regulation only, there is no monetary cost, hence the entry cutoff is monotonically increasing with the search time. Under other policies, the monetary cost contributes to the cutoff. The constant price deters a substantial fraction of motorists from entering at all times of the day, while the occupancy target and the optimal regulation affect mostly those appearing during the congestion period. We can indeed see that good regulation methods (occupancy target and optimal) allow more entry during peak demand hours than other methods. The optimal entry value cutoff is slightly lower than that of the occupancy target around 9:00, allowing a greater proportion of (a large number of) motorists to enter; at the same time, a slightly smaller proportion of (a small number of) motorists enters shortly before 13:00.

Fig. 6 demonstrates the evolution of prices, wherever applicable. As discussed earlier, the optimal price is higher than the occupancy-target price at the beginning of high-entry period, creating more vacancy for those who enter. The optimal price skyrockets during the period when exit is already zero while entry is still positive (between 12:25 and 13:00), reaching the maximum of \$20.67: this limits the number of motorists willing to stay beyond this period.

Fig. 7 demonstrates the generation of social welfare, i.e., value from parking minus the search cost, per parking site, over time. Without regulation, there is a "trough of misery" at 9:00 due to extremely high cost of search for parking. The time-limit policy reduces the depth of the trough, at the cost of eliminating all welfare generation for most of non-arrivals period. Under the constant price regime, the trough of misery is much smaller though still visible; there are some welfare losses during low-demand period due to premature departure of the parkers. The occupancy target and optimal value generation are nearly indistinguishable on Fig. 7; I "zoom in" the difference on Fig. 8. There are two intervals of time when the optimal policy outperforms the occupancy target policy: near peak arrivals at 9:00, due to shorter time of search for parking, and near the end of arrivals at 13:00, by allowing parkers to stay longer and thus to generate more value from parking.

6. Conclusion

This paper develops a policy of pricing parking that formalizes the intuition of congestion pricing by Vickrey (1954). It is shown that the optimal policy differs from the occupancy target policy currently being implemented in San Francisco and in Los Angeles: the price should depend not only on the occupancy but also on the arrivals rate. Although the welfare gains from the new policy may be small (in the proposed numerical example, the instantaneous welfare gains do not exceed 3%,



Fig. 6. Price over time, under various regulation policies. Peak optimal price is \$20.67.



Fig. 7. Social value per parking site over time, under various regulation policies.

and gains aggregated over time are about 0.3%), it should be emphasized that they can be achieved at no additional cost. The hardware used to calculate parking occupancy can be just as easily used to calculate the arrivals rate; the newly proposed formula for parking rate is as simple as the one currently used. The welfare gains from the new policy are especially pronounced at times when the occupancy target (i) becomes binding for the first time and (ii) is no longer binding for the first time. In the first case, the optimal policy prescribes a lower occupancy, making it easier to find a vacant spot for a large number of newly arriving motorists. In the second case, it prescribes a higher occupancy, allowing already present parkers to stay longer and to get more value from their parking session.

The model of demand for parking developed within the paper is general enough and can be used for modeling and comparison of other parking regulation policies.



Fig. 8. Social value of occupancy target policy relative to optimal policy, per parking site over time.

The potential extensions of the model include the following. First, one could account for negative externalities of the search for parking ("cruising"): those who search slow down other traffic. A time-varying positive entry fee may help to internalize the externality.

Second, by assuming a continuum of motorists, the current version removes all stochasticity from the model. At the same time, demand for parking in a specific geographic location can be indeed discrete and therefore stochastic. An inhomogehous Poisson process could be employed to model the phenomenon. In case of discrete demand, there are two possible pricing modes: (i) deterministic pricing, when the price is determined ex-ante and does not respond to actual occupancy, and (ii) stochastic pricing, when the price is adjusted after every parking arrival/departure. In the latter case, one has to account for the fact that each user has a positive impact on the aggregate demand and on the price, thus some aspects of incentive compatibility of the optimal mechanism have to be considered.

Appendix A. Computational details of the numerical example

A1. Preliminary analysis

While the model is complex enough so the equilibrium has to be computed, some results can be derived analytically. I now conduct such analysis for each of the above mentioned regulation scenarios.

A1.1. No regulation

Without regulation, parkers stay until their marginal value is zero. From (20), it follows that $\tau^+(t, \nu) = \tau^-(t', \nu) = \frac{\nu}{s}$, $\forall t, t'$. The utility from a parking session, gross of search costs, is $\int_0^{\tau^+(t,\nu)} u(\tau, \nu) d\tau = \frac{\nu^2}{2s}$. A motorist appearing at time t initiates the search process iff her value from parking exceeds the search cost: $\frac{\nu^2}{2s} \ge \frac{c}{r(1-q(t))}$, which allows us to define the entry cutoff type as follows:

$$v^{E}(t) = \left(\frac{2sc}{r}\right)^{\frac{1}{2}} (1 - q(t))^{-\frac{1}{2}}.$$
(A.1)

As I demonstrate below, in equilibrium q(t) < 1, $\forall t$, and therefore $v^{E}(t)$ is defined for all \$t\$.

Observe that a parker who appeared at $t - \tau$ is still present at t if (i) she has indeed entered at $t - \tau$, i.e., if $v \ge v^{E}(t - \tau)$, and (ii) she did not exit by time t, i.e., if $v \ge v^{X}(\tau)$, where

$$v^{X}(\tau) \equiv s\tau \tag{A.2}$$

is referred to as the *exit cutoff*.

Define by $\tau^*(t)$ the smallest τ such that

$$v^{\mathcal{E}}(t-\tau) - v^{\mathcal{X}}(\tau) \le 0, \forall \tau \ge \tau^*(t). \tag{A.3}$$

Such $\tau^*(t)$ exists because $v^E(t)$ is cyclical, i.e., $v^E(t - nT) = v^E(t)$, $\forall n = 0, 1, 2, ...,$ and is therefore constrained from above, while $v^X(nT)$ is increasing to infinity with n. The motorist types still present at time t are illustrated in Fig. A.9.



Fig. A.9. Determining the types still present at t.

I next modify the expression for q(t) in (5) for better analytical tractability of the latter. Denote $v^{mx}(t, \tau) \equiv \max(v^{E}(t - \tau), v^{X}(\tau))$; the set of motorists entering at $t - \tau$ and still present at t is $[v^{mx}(t, \tau), \infty)$, thus (5) can be rewritten as

$$q(t) = \frac{1}{N} \int_{0}^{\infty} \int_{\nu^{mx}(t,\tau)}^{\infty} A(t-\tau,\nu) d\nu d\tau$$

$$= \frac{1}{N} \int_{0}^{\infty} B(t-\tau) \int_{\nu^{mx}(t,\tau)}^{\infty} \exp\left(-\frac{\nu}{\lambda}\right) d\nu d\tau$$

$$= \frac{1}{N} \int_{0}^{\infty} B(t-\tau) \lambda \exp\left(-\frac{\nu^{mx}(t,\tau)}{\lambda}\right) d\tau$$

$$= \frac{1}{N} \int_{0}^{\tau^{*}(t)} B(t-\tau) \lambda \exp\left(-\frac{\nu^{mx}(t,\tau)}{\lambda}\right) d\tau + \frac{1}{N} \int_{\tau^{*}(t)}^{\infty} B(t-\tau) \lambda \exp\left(-\frac{\nu^{X}(\tau)}{\lambda}\right) d\tau.$$
(A.4)

Using the definition of $v^X(\tau)$ and the fact that $B(t) = B(t+T), \forall t$, we can modify the last component of (A.4) as follows:

$$X(t) = \frac{1}{N} \int_{\tau^{*}(t)}^{\infty} B(t-\tau)\lambda \exp\left(-\frac{\nu^{X}(\tau)}{\lambda}\right) d\tau$$

$$= \frac{1}{N} \exp\left(-\frac{st}{\lambda}\right) \int_{\tau^{*}(t)}^{\infty} B(t-\tau)\lambda \exp\left(\frac{s(t-\tau)}{\lambda}\right) d\tau$$

$$= \frac{1}{N} \exp\left(-\frac{st}{\lambda}\right) \int_{-\infty}^{t-\tau^{*}(t)} B(t')\lambda \exp\left(\frac{st'}{\lambda}\right) dt'$$

$$= \frac{1}{N} \frac{\exp\left(-\frac{st}{\lambda}\right)}{1-\exp\left(-\frac{st}{\lambda}\right)} \int_{t-\tau^{*}(t)-T}^{t-\tau^{*}(t)} B(t')\lambda \exp\left(\frac{st'}{\lambda}\right) dt'.$$
(A.5)

Also denote

$$E(t,\tau) \equiv B(t-\tau)\lambda \exp\left(-\frac{\nu^{\max}(t,\tau)}{\lambda}\right)$$
(A.6)

the mass of parkers who entered at $t - \tau$ and still present at *t*. With new notations, we can rewrite (A.4) as follows:

$$q(t) = \frac{1}{N} \int_0^{\tau^*(t)} E(t, \tau) d\tau + X(t).$$
(A.7)

Likewise, we can modify (6) as follows. Its double integral with respect to τ and ν can be rewritten as

$$\int_{\mathcal{V}} \int_{0}^{\tau^{-}(t,v)} I(t-\tau,v)A(t-\tau,v)u(\tau,v)d\tau dv$$

$$= \int_{0}^{\infty} B(t-\tau) \int_{v^{mx}(t,\tau)}^{\infty} (v-s\tau) \exp\left(-\frac{v}{\lambda}\right) dv d\tau$$

$$= \int_{0}^{\infty} B(t-\tau) (v^{mx}(t,\tau) - s\tau + \lambda)\lambda \exp\left(-\frac{v^{mx}(t,\tau)}{\lambda}\right) d\tau$$

$$= \int_{0}^{\tau^{*}(t)} (v^{mx}(t,\tau) - s\tau + \lambda)E(t,\tau) d\tau + \lambda X(t).$$
(A.8)

The search cost at time t in (6) can be rewritten as

$$\frac{c}{r}\frac{1}{1-q(t)}\int_{\mathcal{V}}I(t,\nu)A(t,\nu)d\nu = \frac{c}{r}\frac{E(t,0)}{1-q(t)}$$
(A.9)

This allows us do redefine (6) as follows:

$$V = \int_0^T \int_0^{\tau^*(t)} (v^{\max}(t,\tau) - s\tau + \lambda) E(t,\tau) d\tau + \lambda N X(t) - \frac{c}{r} \frac{E(t,0)}{1 - q(t)} dt.$$
(A.10)

A1.2. Constant price

The parking authority sets a constant price *P* per unit of parking time. From (20), it follows that $\tau^+(t, v) = \tau^-(t', v) = \frac{v-P}{s}$, $\forall t, t'$. The utility from a parking session, gross of search costs, is $\int_0^{\tau^+(t,v)} (u(\tau, v) - P)d\tau = \frac{(v-P)^2}{2s}$. As before, entry decisions are made by comparing the gross utility with the search cost (cf.(A.1)):

$$v^{E}(t) = P + \left(\frac{2sc}{r}\right)^{\frac{1}{2}} (1 - q(t))^{-\frac{1}{2}}.$$
(A.11)

The exit cutoff is (cf.(A.2)) $v^X(\tau) = P + s\tau$; the definitions of $\tau^*(t)$, $E(t, \tau)$ are unchanged from (A.3,A.6), respectively. The occupancy at time *t* is then (cf.(A.7))

$$q(t) = \frac{1}{N} \int_0^{\tau^*(t)} E(t,\tau) d\tau + \exp\left(-\frac{P}{\lambda}\right) X(t),$$
(A.12)

where X(t) is unchanged from (A.5). The social welfare is redefined as (cf.(A.10))

$$V = \int_0^T \int_0^{\tau^*(t)} \left(\nu^{\max}(t,\tau) - s\tau + \lambda \right) E(t,\tau) d\tau + (P+\lambda) N \exp\left(-\frac{P}{\lambda}\right) X(t) - \frac{c}{r} \frac{E(t,0)}{1-q(t)} dt.$$
(A.13)

Note that (A.13) adds up motorists' welfare and the revenue of the parking authority. The price *P* is chosen to maximize (A.13).

A1.3. Time limit

The parking authority sets a time limit $\bar{\tau}$ on the duration of parking. From (20), it follows that $\tau^+(t, \nu) = \tau^-(t', \nu) = \min\{\frac{\nu}{\tau}, \bar{\tau}\}, \forall t, t'$. The utility from a parking session, gross of search costs, is

$$\int_0^{\tau^+(t,\nu)} u(\tau,\nu) \mathrm{d}\tau = \begin{cases} \frac{\nu^2}{2s}, & \nu < s\bar{\tau}, \\ \nu\bar{\tau} - s\frac{\bar{\tau}^2}{2}, & \nu \ge s\bar{\tau} \end{cases}$$
(A.14)

If the occupancy rate is low enough, the entry cutoff $v^{E}(t)$ is below $s\bar{\tau}$, meaning that some parkers entering at t will depart before the time limit expires. Specifically, if $q(t) < \bar{q} \equiv 1 - 2\frac{c}{rs\bar{\tau}^{2}}$, then $v^{E}(t)$ is described by (A.1). Otherwise, the entry cutoff is above or equal to $s\bar{\tau}$, meaning that all parkers entering at t will stay until the end of the time limit. In algebra, if $q(t) \ge \bar{q}$, then $v^{E}(t) = \frac{c}{r\bar{\tau}(1-q(t))} + \frac{1}{2}s\bar{\tau}$. The equation for occupancy is (cf.(A.7))

$$q(t) = \frac{1}{N} \int_0^{\min\{\tau^*(t),\bar{\tau}\}} E(t,\tau) d\tau + \bar{X}(t),$$
(A.15)

where $\tau^*(t)$, $E(t, \tau)$ are defined in (A.3,A.6), respectively. If $\tau^*(t) \ge \overline{\tau}$, then $\overline{X}(t) \equiv 0$; otherwise (cf.(A.5))

$$\bar{X}(t) = \frac{1}{N} \int_{\tau^*(t)}^{\bar{\tau}} B(t-\tau) \lambda \exp\left(-\frac{\nu^X(\tau)}{\lambda}\right) d\tau = \frac{1}{N} \exp\left(-\frac{st}{\lambda}\right) \int_{t-\bar{\tau}}^{t-\tau^*(t)} B(t') \lambda \exp\left(\frac{st'}{\lambda}\right) dt'.$$
(A.16)

The social welfare can be computed as (cf.(A.10))

$$V = \int_0^T \int_0^{\min\{\tau^*(t),\bar{\tau}\}} (v^{\max}(t,\tau) - s\tau + \lambda) E(t,\tau) d\tau + \bar{X}(t) - \frac{c}{r} \frac{E(t,0)}{1 - q(t)} dt.$$
(A.17)

A1.4. Occupancy target

The parking authority targets an occupancy rate $Q \in (0, 1)$ by introducing a price function p(t). The duration of parking is $\tau^{-}(t', \nu) = \frac{\nu - p(t')}{s}$, and thus

$$\tau^{+}(t,v) = \frac{v - p(t + \tau^{+}(t,v))}{s}.$$
(A.18)

The utility from a parking session, gross of search costs, is $\int_0^{\tau^+(t,v)} (u(\tau,v) - p(t+\tau))d\tau = v\tau^+(t,v) - \frac{s}{2}\tau^+(t,v)^2 - \int_0^{\tau^+(t,v)} p(t+\tau)d\tau$. The entry cutoff $v^E(t)$ and the minimal duration of stay $\tau^E(t) \equiv \tau^+(t,v^E(t))$ satisfy the system of equations (cf.(A.18))

$$\nu^{E}(t)\tau^{E}(t) - \frac{s}{2}\tau^{E}(t)^{2} - \int_{0}^{\tau^{E}(t)} p(t+\tau)d\tau \equiv \frac{c}{r}\frac{1}{1-q(t)},$$
(A.19)

$$\tau^{E}(t) = \frac{\nu^{E}(t) - p(t + \tau^{E}(t))}{s}.$$
(A.20)

The exit cutoff is (cf.(A.2)) $v^X(t, \tau) = p(t) + s\tau$; the functions $\tau^*(t)$, $E(t, \tau)$ are unchanged from (A.3,A.6). The occupancy at t satisfies (cf.(A.12))

$$q(t) = \frac{1}{N} \int_0^{\tau^*(t)} E(t,\tau) d\tau + \exp\left(-\frac{p(t)}{\lambda}\right) X(t),$$
(A.21)

with X(t) unchanged from (A.5). The price p(t) is such that

$$q(t) \begin{cases} \leq Q, & p(t) = 0, \\ = Q, & p(t) > 0. \end{cases}$$
(A.22)

The social value is given by (cf.(A.13))

$$V = \int_0^T \int_0^{\tau^*(t)} \left(\nu^{\max}(t,\tau) - s\tau + \lambda \right) E(t,\tau) d\tau + (p(t) + \lambda) N \exp\left(-\frac{p(t)}{\lambda}\right) X(t) - \frac{c}{r} \frac{E(t,0)}{1 - q(t)} dt.$$
(A.23)

The occupancy target Q is chosen to maximize welfare (A.23).

If the occupancy constraint is still binding, and therefore the price still positive, at the end of the arrivals period ($t = t^{end}$, equal to 13 in our example), then pricing beyond this period is worth special attention.

Lemma 1. If $p(t^{\text{end}}) > 0$, then the authority must charge an 'exit fee' of $F^{\text{ex}} \equiv \frac{1}{2s}p(t^{\text{end}})^2$ for all those willing to stay beyond t^{end} . In such case, there will be no departures within $[t^{\text{end}}, t^{\text{end}} + \frac{1}{s}p(t^{\text{end}})]$.

Note that the one-time exit fee specified above is equivalent to continuously paying the price falling from $p(t^{end})$ to zero at a linear rate of *s* beyond the arrivals period:

$$F^{\text{ex}} = \int_{t^{\text{end}}}^{t^{\text{end}} + \frac{1}{s}p(t^{\text{end}})} p(t^{\text{end}}) - s(t' - t^{\text{end}}) dt'.$$

In the illustrations below (Fig. 6), I show the price decreasing in this way beyond t^{end} .

Proof. First, suppose the exit fee is smaller: $F^{\text{ex}} < \frac{1}{2s}p(t^{\text{end}})^2$. Consider the lowest-marginal-value parker still present at time $t^{\text{end}} - dt$, with dt being small. Since parkers stay until their value equals the price, the remaining value v^{\min} of such marginal parker is no less than $p(t^{\text{end}} - dt) \approx p(t^{\text{end}})$. The utility of such parker from staying until marginal value becomes zero, i.e., until $t^{\text{end}} - dt + \frac{1}{s}v^{\min} > t^{\text{end}}$, is equal to

$$\frac{1}{2s}(v^{\min})^2 - p(t^{\text{end}})\mathrm{d}t - F^{\text{ex}},$$

which is positive and is greater than the utility from exiting anytime before t^{end} , for a sufficiently small dt. Therefore, the marginal parker at time $t^{\text{end}} - dt$, and thus all other parkers at that time, will choose to stay until after t^{end} . The same conclusion applies to everyone present within $[t^{\text{end}} - dt, t^{\text{end}}]$, and thus there will be no departures during this period. At the same time, the arrivals rate is still positive within this interval, meaning that the occupancy must be strictly increasing over time. On the other hand, p(t) > 0 on this interval implies that the occupancy is at its maximal level and is therefore constant, which is a contradiction.

Second, suppose the exit fee is above the specified value: $F^{\text{ex}} > \frac{1}{2s}p(t^{\text{end}})^2$. We can show that in this case a strictly positive mass of parkers will choose to exit exactly at t^{end} , in order to avoid paying the fee, and the occupancy would discontinuously drop below the constraint. This would contradict the rules of the occupancy target policy, that the fees collected are just high enough to meet the occupancy constraint. \Box

A1.5. Optimal policy

The optimal policy is similar to the occupancy target policy. One difference is that the price is determined by (9) rather than by (A.22). The definition (9) can be rewritten, using the notations of the Appendix, as follows:

$$p(t) = \frac{c}{rN} \frac{E(t,0)}{(1-q(t))^2}.$$
(A.24)



Fig. A.10. Division of parkers into short-term parkers and long-term parkers.

Another important difference is the behavior of parkers shortly before the end of the arrivals period. By (A.24), parking must be free after t^{end} ; at the same time, the price may fall sharply enough just before t^{end} so that some parkers have their utility double-peaked with respect to exit time. In this case, some parkers may choose to stay even when their marginal value is below the current price, because they expect a sharp price drop in the near future. For this reason, all parkers are divided into (i) the *short-term parkers*, who exit before t^{end} , when their marginal value is equated to the price, and (ii) the *long-term parkers*, who stay until after t^{end} , as long as the marginal value is positive. The cutoff type is indifferent between a short stay and a long stay, meaning that their additional integrated value from a longer stay is exactly equal to the additional monetary payment for the longer stay.

Fig. A.10 illustrates the cutoff line between the short-term parkers and the long-term parkers. The slope of the line is -s, i.e., the rate of decay of everyone's instantaneous value, thus every parker is either always below or always above the cutoff line. The position of the cutoff line is determined by the condition that area A is equal to area B. The point t_2 is the intersection of the price curve and the cutoff line; it is also the time of departure of the last short-term parker.

Note that the motorists entering the model during the no-departure period (i.e., within [t_2 , t^{end}]) are all long-term parkers, yet some of them may be below the cutoff line, meaning that they exit before all those long-term parkers who arrived before t_2 .

Given the fact that parkers do not always exit when their instantaneous value drops below the price, we need to redefine the function $v^{X}(t, \tau)$ for $t > t_2$. Parkers that arrived before t_2 still stay at t if their instantaneous value at t is positive and is above the cutoff line shown on Fig. A.10. Parkers that arrived after t_2 are still present if their instantaneous value at t is positive. In math,

$$v^{X}(t,\tau) = \begin{cases} p(t) + s\tau, & t \le t_{2} \\ p(t_{2}) + s\tau - s(t-t_{2}), & t \in \left[t_{2}, t_{2} + \frac{p(t_{2})}{s}\right] \& \tau \ge t - t_{2} \\ s\tau, & \text{otherwise} \end{cases}$$
(A.25)

A2. Computational details

The general philosophy of searching for an equilibrium is to assume some initial values of model parameters; given these values, calculate all model ingredients, including the new values of the same parameters. Equating the initial values to the new values of the model parameters constitutes a system of nonlinear equations which is then solved numerically. To facilitate the numerical search of the solution, I also calculate the derivatives of all model ingredients with respect to initially assumed values of model parameters, so the Jacobian matrix for the system of equations is known. I now proceed to describing the details.

A2.1. Initially assumed parameters

I assume some initial values of the entry cutoff $v^{E}(t)$, labeled $v_{0}^{E}(t)$, on a grid of time points. The grid is limited to the arrivals period (i.e., from $t \in [5, 13]$) and is equally spaced with 1 min time interval, resulting in 481 grid points. In case of the optimal regime, I also assume an initial value of t_{2} , the time when the last short-term parker quits.

A2.2. Calculation of occupancy and price

I evaluate q(t), and p(t) if applicable, on the same time grid. To find q(t) at a given time t (with a given value of p(t) if applicable), I identify all time lags τ at which $v_0^E(t - \tau)$ and $v^X(t, \tau)$ intersect; since such intersections typically do not coincide with the grid points, I evaluate them by assuming that $v_0^E(\cdot)$ evolves linearly between the grid points. For most accurate results, I insert such points of intersection of $v_0^E(t - \tau)$ and $v_X(t, \tau)$, for a given t, as auxiliary time grid points before evaluating the integrals used to calculate q(t).

In case of the occupancy target regime, for every time point on the grid I evaluate two parameters: (i) occupancy q(t) in case of p(t) = 0, and (ii) price p(t) that equates occupancy to the target (q(t) = Q). Although at a typical point in time, only one of the two values is relevant, knowing both is useful because it allows us to infer the precise time of switching from non-binding to binding occupancy target, i.e., the time at which both p(t) = 0 and q(t) = Q are true. I add such switching time as an auxiliary time grid point, which later helps us to evaluate more accurately the expected monetary costs for newly arriving motorists.

In case of optimal regime, if $t \le t_2$, I evaluate q(t) and p(t) simultaneously to meet the constraint (A.24). If $t > t_2$, parkers no longer respond to the price, therefore I calculate the occupancy q(t) first, and then use (A.24) to infer p(t). When evaluating q(t), I pay attention to the fact that $v^X(t, \tau)$ is discontinuous at $\tau = t - t_2$ if $t \in [t_2, t_2 + \frac{p(t_2)}{s}]$; I insert an auxiliary time grid point.

A2.3. Calculation of entry cutoff

In case of no regulation, constant price, time limit regimes the new values of $v^{E}(t)$ can be found analytically once q(t) is known. Since the value of q(t) calculated above is not guaranteed to be less than unity, which is necessary to find a meaningful value of $v^{E}(t)$, I impose an upper bound on q(t).

In case of the occupancy target regime, the cutoff entry value $v^{E}(t)$ and the minimal duration of stay $\tau^{E}(t)$ are jointly determined from the system of equations (A.19,A.20). As implied by earlier discussion, there exists a discontinuity in $\tau^{E}(t)$ (but not in $v^{E}(t)$) at some point $t_{1} < t^{\text{end}}$, due to a sharp drop in price after t^{end} and resulting zero departures between t^{end} and $t^{\text{end}} + \frac{1}{s}p(t^{\text{end}})$. This time point satisfies

$$\begin{split} &\lim_{t \neq t_1} t + \tau^E(t) = t^{\text{end}}, \\ &\lim_{t \searrow t_1} t + \tau^E(t) = t^{\text{end}} + \frac{1}{s} p(t^{\text{end}}), \\ &\tau^E(t) = v^E(t)/s, t > t_1. \end{split}$$

Under the optimal regime, there also exists a no-departure period of time, but unlike the occupancy target regime, the beginning of such period t_2 is not ex-ante known. For this reason, at each point in time I calculate two sets of the cutoff entry value $v^E(t)$ and duration of stay $\tau^E(t)$:

- (i) Short stay: parkers exit when marginal value is equated to price for the first time. The unknowns $v_{SS}^{E}(t)$, $\tau_{SS}^{E}(t)$ are found from the system (A.19,A.20).
- (ii) Long stay: parkers exit when marginal value equals zero. The unknowns $v_{LS}^E(t)$, $\tau_{LS}^E(t)$ are found from the system (A.19) and $\tau_{LS}^E(t) = \frac{1}{5}v_{LS}^E(t)$.

The entry cutoff value is the minimum of the two: $v^E(t) = \min(v^E_{SS}(t), v^E_{LS}(t))$. I also calculate the new commencement time of the no-departures period, t_2 , as follows. I first find t_1 , the entry time of the last short-term parker: it is the earliest time that satisfies $v^E_{SS}(t_1) = v^E_{LS}(t_1)$. The exit time of the last short-term parker then satisfies $t_2 = t_1 + \tau^E_{SS}(t_1)$.

A2.4. Additional parameters of interest

Having found the equilibrium, I calculate the parameters of interest, some of which (occupancy, social value) are calculated not only within the arrivals period but also at other times of the day. For those other times ($t \in [0, 5], t \in [13, 24]$), I introduce an additional time grid with equally spaced 3 min intervals. In the constant price and occupancy target scenarios I choose the value of a parameter (P, Q, respectively) by maximizing the social welfare integrated over time.

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